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Instanton Calculus in R-R 3-form Background and Deformed $\mathcal{N}=2$ Super Yang-Mills Theory

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Abstract

We study the ADHM construction of instantons in $\mathcal{N}=2$ supersymmetric Yang-Mills theory deformed in constant Ramond-Ramond (R-R) 3-form field strength background in type IIB superstrings. We compare the deformed instanton effective action with the effective action of fractional D3/D(-1) branes at the orbifold singularity of $\mathbf{C}^2/\mathbf{Z}_2$ in the same R-R background. We find discrepancy between them at the second order in deformation parameters, which comes from the coupling of the translational zero modes of the D(-1)-branes to the R-R background. We improve the deformed action by adding a term with space-time dependent gauge coupling. Although the space-time action differs from the action in the Ω-background, both actions lead to the same instanton equations of motion at the lowest order in gauge coupling.

Contents

1 Introduction

Closed string background in superstring theories induces non-trivial effects on D-branes, which are useful to study non-perturbative properties in supersymmetric gauge theories. For example, constant NS-NS B-fields along D-branes induce noncommutativity on the world-volume [1, 2]. Noncommutative instanton [3, 4] is a basic object for studying the ADHM moduli space of instantons [5], which resolves small instanton singularity.

Closed Ramond-Ramond (R-R) backgrounds also bring novel effects on the D-branes. In fact, constant self-dual graviphoton backgrounds are utilized to investigate F-terms in supersymmetric gauge theories via closed/open string duality [6, 7, 8, 9]. In this set-up, it is important to fix the scaling condition for the (self-dual) graviphoton field strength $\mathcal{F}_{\alpha\beta}$, where α, β are spinor indices in four-dimensional space-time. For example, in [9] the zero slope limit $\alpha' \to 0$ with fixed $(2\pi\alpha')^{-\frac{1}{2}}\mathcal{F}_{\alpha\beta}$ was considered. On the other hand, the self-dual graviphoton background $\mathcal{F}_{\alpha\beta}$ with fixed $(2\pi\alpha')^{\frac{3}{2}}\mathcal{F}_{\alpha\beta}$ provides a non(anti)commutative

deformation of $\mathcal{N}=1$ superspace [10, 11, 12]. $\mathcal{N}=1$ supersymmetric gauge theories in non(anti)commutative superspace has been studied extensively (see [12, 13] for example). The instanton solution and its moduli space are also deformed by non(anti)commutativity [14, 15]. In [15], the instanton is realized in D3/D(-1)-brane system at the singularity on the orbifold $\mathbf{R}^6/\mathbf{Z}_2 \times \mathbf{Z}_2$ in the graviphoton background.

Non(anti)commutative $\mathcal{N}=1$ superspace is generalized to $\mathcal{N}=2$ extended superspace, which admits the singlet and non-singlet type of deformations [16]. Supersymmetric gauge theory on non(anti)commutative $\mathcal{N}=2$ harmonic superspace [16, 17, 18, 19, 20] can be realized on D3-branes at the singularity in the orbifold $\mathbf{C}^2/\mathbf{Z}_2$ in the R-R 5-form background $\mathcal{F}^{\alpha\beta IJ}$ with the same scaling condition as in $\mathcal{N}=1$ non(anti)commutative case. Here $I,J=1,2$ are $SU(2)_R$ R-symmetry indices. It has been shown in [21] that symmetric-symmetric (S,S) type field strength $\mathcal{F}^{(\alpha\beta)(IJ)}$ corresponds to the non-singlet deformation. An antisymmetric-antisymmetric (A,A) type field strength $\mathcal{F}^{[\alpha\beta][IJ]}$ is expected to correspond to the singlet deformation [17]. The deformed instanton equation with some special deformation parameters was discussed in [22]. Prepotential of non(anti)commutative gauge theory with singlet deformation was also discussed in [20]. Using string theory technique, further extension to the $\mathcal{N}=4$ gauge theory in the R-R 5-form graviphoton background was investigated [23], but their instanton solutions are not yet studied so far. Recently its gravity dual has been proposed in [24].

In superstring theory there are R-R backgrounds with various rank. In type IIB theory, for example, there are R-R 3-forms (and its dual), which correspond to the backgrounds $\mathcal{F}^{(\alpha\beta)[AB]}$ and $\mathcal{F}^{[\alpha\beta](AB)}$, denoted as (S,A) and (A,S) type deformations [23]. Here $A,B=1,\cdots,4$ are $SU(4)_R$ R-symmetry indices. By orbifolding $\mathbf{C}^2/\mathbf{Z}_2$, we can introduce deformation of $\mathcal{N}=2$ theory. For the (S,A)-type deformation, the field strengths become $\mathcal{F}^{(\alpha\beta)(IJ)}$ and $\mathcal{F}^{(\alpha\beta)[I'J']}$ $(I',J'=3,4)$. These deformations cannot be realized in terms of non(anti)commutative superspace, but have interesting non-perturbative effects.

Recently, in [25] the low-energy effective action of a system of fractional D3 and D(-1)-branes was studied in the (S,A)-type background with fixed $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[IJ]}$ and $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[I'J']}$. They observed that the effective action of a fractional D3/D(-1) system agrees with the instanton effective actions of gauge theory in the Ω-background [26] by identifying the R-R 3-form field strengths with the Ω-background. The instanton effective

action in the Ω background plays an important role to obtain the closed form of the prepotential in $\mathcal{N}=2$ supersymmetric gauge theory with help of the localization technique [26, 27, 28, 29, 30]. Since the fractional D3/D(-1) system in the R-R 3-form background provides a simple string setup, it is important to study the relation between the R-R 3-form background and the Ω-background in viewpoint of application to more general system. In a previous paper [31], we studied deformation of $\mathcal{N}=2$ and 4 super Yang-Mills theories in the (S,A) or (A,S) type R-R 3-form background [1]. It would be natural to expect that the deformed $\mathcal{N}=2$ gauge theory gives the effective action of D(-1)-branes in the R-R background. However, there are some subtleties to identify both theories. The deformed action of the D3-branes in the R-R 3-form background is rather different from that of gauge theory in the Ω-background. Gauge theories deformed in the constant R-R 3-form background have manifest translational symmetry, but the Ω-background metric contains space-time coordinates explicitly and translational invariance is lost.

The aim of this paper is to study the relation between $\mathcal{N}=2$ super Yang-Mills theory deformed in the R-R 3-form background and the fractional D3/D(-1) effective action. We will solve the instanton equations in deformed theory using the ADHM construction [34] up to the second order in the deformation parameter. We then construct the instanton effective action from the field theory and compare it with that obtained from the string theory. We will see that discrepancy arises at the second order in the deformation parameter, which comes from the absence of coupling of translational zero modes to the R-R background in the gauge theory side. When we want to reproduce this coupling as an instanton solution, we need to add one term to the deformed action at the second order. The improved action has the space-time dependent gauge coupling, which is similar to that in the Ω-background. But two actions are shown to be different. However, they have the same instanton equations of motion at the lowest order in gauge coupling and give the same instanton effective action.

This analysis can be generalized into the $\mathcal{N}=4$ super Yang-Mills theory in the R-R 3-form background and $\mathcal{N}=4$ version of the Ω-background. This subject will be discussed in the next paper [32].

The organization of this paper is as follows. In section 2, we introduce four-dimensional

[1] In [25] the $\mathcal{N}=2$ deformed Lagrangian with $\mathcal{F}^{(\alpha\beta)[I'J']}=0$ was obtained.

(S,A)-deformed $\mathcal{N}=2$ $U(N)$ super Yang-Mills action defined on (fractional) D3-branes at the singularity of the orbifold $\mathbf{C}^2/\mathbf{Z}_2$. The instanton equation is obtained and solved via the ADHM construction. We calculate the instanton effective action for the self-dual solution and compare this result with the D3/D(−1)-branes result [25]. However, once we introduce a term which breaks translational symmetry of the deformed action, both results agree even at the second order.

In section 3, the relation between the (S,A)-deformed super Yang-Mills theory and the Ω-background is discussed. Section 4 is devoted to conclusions and discussions. We make a comment on the mass term deformation of the instanton effective action induced by the (A,S)-type background. A brief introduction to the ADHM construction of instantons is presented in appendix A. A detailed calculation of the instanton effective action can be found in appendix B.

2 Instanton calculus in the (S,A)-deformed $\mathcal{N}=2$ super Yang-Mills theory

In this section, we discuss four-dimensional $\mathcal{N}=2$ $U(N)$ super Yang-Mills theory deformed by the (S,A)-type R-R 3-form background [31] and calculate the instanton solution and the instanton effective action. $\mathcal{N}=2$ $U(N)$ super Yang-Mills theory is described by gauge fields A_μ ($\mu=1,2,3,4$), complex scalars φ, $\bar{\varphi}$ and Weyl fermions Λ^I_α and $\bar{\Lambda}^{\dot{\alpha}}_I$ ($I=1,2$), which belong to the adjoint representation of gauge group $U(N)$. We denote T^m as the basis of $U(N)$ generators normalized as $\mathrm{Tr}(T^mT^n)=\kappa\delta^{mn}$ with constant κ. The Lagrangian is given by

$$\mathcal{L}_0=\frac{1}{\kappa}\mathrm{Tr}\Bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i\theta g^2}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}-D_\mu\varphi D^\mu\bar{\varphi}-\frac{1}{2}g^2[\varphi,\bar{\varphi}]^2$$
$$-i\Lambda^{I\alpha}(\sigma^\mu)_{\alpha\dot{\beta}}D_\mu\bar{\Lambda}^{\dot{\beta}}_I+\frac{i}{\sqrt{2}}g\,\Lambda^I[\bar{\varphi},\Lambda_I]-\frac{i}{\sqrt{2}}g\,\bar{\Lambda}_I[\varphi,\bar{\Lambda}^I]\Bigg]. \tag{2.1}$$

Here $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu,A_\nu]$ is the gauge field strength, g is the gauge coupling constant and $D_\mu * = \partial_\mu * + ig[A_\mu, *]$ is a gauge covariant derivative. We also define $\sigma_\mu=(i\tau^1,i\tau^2,i\tau^3,1)$ and $\bar{\sigma}_\mu=(-i\tau^1,-i\tau^2,-i\tau^3,1)$, where τ^i ($i=1,2,3$) are the Pauli matrices. θ is a theta angle and $\tilde{F}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}$. This theory is the low-energy effective

theory of N (fractional) D3-branes on $\mathbf{C}\times\mathbf{C}^2/\mathbf{Z}_2$, where the D3-branes are located in the fixed point of the orbifold [33].

We now introduce the (S,A)-type R-R 3-form $\mathcal{F}^{(\alpha\beta)[AB]}$. After $\mathbf{Z}_2$ orbifolding, the surviving components are $\mathcal{F}^{(\alpha\beta)12}$ and $\mathcal{F}^{(\alpha\beta)34}$, from which we define $\mathcal{N}=2$ deformation parameters as $C^{\alpha\beta}=4\sqrt{2}\pi(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)12}$, $\bar{C}^{\alpha\beta}=4\sqrt{2}\pi(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)34}$.

We also use the notation $C^{\mu\nu}\equiv\varepsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma C^{\alpha\beta}$ and $\bar{C}^{\mu\nu}\equiv\varepsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma\bar{C}^{\alpha\beta}$ where $\sigma^{\mu\nu}=\frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)$. $C^{\mu\nu}$ corresponds to the self-dual graviphoton field strength in $\mathcal{N}=2$ supergravity multiplet while $\bar{C}^{\mu\nu}$ corresponds to the self-dual background of the vector multiplet [25]. The deformed Lagrangian up to the second order in the deformation parameter is [31]

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_C, \tag{2.2}$$

where the second term $\mathcal{L}_C$ in (2.2) is the interaction term obtained from the computation of disk amplitudes of open strings in the R-R 3-form background;

$$\mathcal{L}_C=\frac{1}{\kappa}\mathrm{Tr}\left[ig(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)F_{\mu\nu}+\frac{i}{\sqrt{2}}g\Lambda_\alpha{}^I\Lambda_{\beta I}\bar{C}^{(\alpha\beta)}+\frac{1}{2}g^2(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)^2\right]. \tag{2.3}$$

We study the instanton solution of the deformed theory based on the Euclidean action. The bosonic part relevant to the gauge instanton is written in the perfect square form S' as

$$\begin{aligned}S'&=\int d^4x\,\frac{1}{\kappa}\mathrm{Tr}\left[\frac{1}{2}\big(F^{(+)}_{\mu\nu}-ig(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)\big)^2\right]+\left(-\frac{8\pi^2}{g^2}+i\theta\right)k\\ &=\int d^4x\,\frac{1}{\kappa}\mathrm{Tr}\left[\frac{1}{2}\big(F^{(-)}_{\mu\nu}\big)^2-ig(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)F^{(+)}_{\mu\nu}-\frac{g^2}{2}(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)^2\right]+\left(\frac{8\pi^2}{g^2}+i\theta\right)k.\end{aligned} \tag{2.4}$$

where $F^{(\pm)}_{\mu\nu}=\frac{1}{2}(F_{\mu\nu}\pm\tilde{F}_{\mu\nu})$. The instanton number k is defined by

$$k=\frac{g^2}{32\pi^2}\int d^4x\,\frac{1}{\kappa}\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2.5}$$

We then obtain the self-dual and the anti-self-dual equations.

$$F^{(-)}_{\mu\nu}=0, \tag{2.6}$$

$$F^{(+)}_{\mu\nu}-ig(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)=0. \tag{2.7}$$

A solution corresponding to the equation (2.6) is called the self-dual solution while the one to the equation (2.7) is the anti-self-dual solution. The other fields satisfy the equation of motion in the (anti-)self-dual background. From the Lagrangian (2.2), the equations of motion are derived as

$$
\begin{aligned}
&D^2\bar{\varphi} - i\sqrt{2}g\bar{\Lambda}_I\bar{\Lambda}^I - g^2\Big[\bar{\varphi},[\varphi,\bar{\varphi}]\Big] + igF_{\mu\nu}\bar{C}^{\mu\nu} + g^2\varphi\bar{C}_{\mu\nu}\bar{C}^{\mu\nu} + g^2\bar{\varphi}C_{\mu\nu}\bar{C}^{\mu\nu} = 0,\\
&D^2\varphi + i\sqrt{2}g\Lambda^I\Lambda_I - g^2\Big[\varphi,[\bar{\varphi},\varphi]\Big] + igF_{\mu\nu}C^{\mu\nu} + g^2\bar{\varphi}C_{\mu\nu}C^{\mu\nu} + g^2\varphi C_{\mu\nu}\bar{C}^{\mu\nu} = 0,\\
&(\sigma^\mu)_{\alpha\dot{\beta}}D_\mu\bar{\Lambda}_I{}^{\dot{\beta}} + \sqrt{2}g[\bar{\varphi},\Lambda_{I\alpha}] + \sqrt{2}g\,\bar{C}_{\alpha\beta}\Lambda_I^\beta = 0,\\
&(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}D_\mu\Lambda^I{}_\beta - \sqrt{2}g[\varphi,\bar{\Lambda}^{I\dot{\alpha}}] = 0,\\
&D_\mu\left(F^{\mu\nu} - 2ig\bar{\varphi}C^{\mu\nu} - 2ig\varphi\bar{C}^{\mu\nu}\right)\\
&\qquad -ig[\varphi,D^\nu\bar{\varphi}] - ig[\bar{\varphi},D^\nu\varphi] - g(\sigma^\nu)_{\alpha\dot{\beta}}\{\Lambda^{I\alpha},\bar{\Lambda}_I{}^{\dot{\beta}}\} = 0.
\end{aligned}
\tag{2.8}
$$

First we consider the case where the vacuum expectation values (VEVs) of the scalar fields are zero. In this case, we find some exact solutions. For example, in the case of $\bar{C}^{\mu\nu} = 0$, the Dirac equation for the fermion $\bar{\Lambda}_{\dot{\alpha}}$ has no zero mode in the self-dual background. We can set $\bar{\Lambda} = 0$. Then the equation of motion for $\bar{\varphi}$ becomes

$$
D^2\bar{\varphi} - g^2\Big[\bar{\varphi},[\varphi,\bar{\varphi}]\Big] = 0, \tag{2.9}
$$

from which $\bar{\varphi} = 0$ is found to be an exact solution. Therefore $\bar{\varphi} = \bar{\Lambda} = 0$ is shown to be an exact solution. Then the equation of motion for the other fields becomes

$$
F^{(-)}_{\mu\nu} = 0, \tag{2.10}
$$

$$
(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}D_\mu\Lambda^I_\beta = 0, \tag{2.11}
$$

$$
D^2\varphi + i\sqrt{2}g\Lambda^I\Lambda_I + iC^{\mu\nu}F_{\mu\nu} = 0. \tag{2.12}
$$

The equations (2.10)–(2.12) are solved by the ADHM construction [34] (see appendix A) for any instanton number k as

$$
A_\mu = -i\bar{U}\partial_\mu U, \tag{2.13}
$$

$$
\Lambda^I_\alpha = \bar{U}(\mathcal{M}^I f\bar{b}_\alpha - b_\alpha f\bar{\mathcal{M}}^I)U, \tag{2.14}
$$

$$
\varphi = -i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{U}\mathcal{M}^I f\bar{\mathcal{M}}^J U + \bar{U}\begin{pmatrix} 0 & 0 \\ 0 & \chi\mathbf{1}_2 + \mathbf{1}_k C \end{pmatrix}U. \tag{2.15}
$$

Here U is the $(N+2k)\times N$ matrix which satisfies $\bar{\Delta}^{\dot{\alpha}}U=0$ with the $(N+2k)\times 2k$ matrix

$$\Delta_{\dot{\alpha}}=a_{\dot{\alpha}}+b^{\beta}(\sigma_{\mu})_{\beta\dot{\alpha}}x^{\mu}=\begin{pmatrix} w_{\dot{\alpha}} \\ (a'+x)_{\alpha\dot{\alpha}}\end{pmatrix},\quad x_{\alpha\dot{\alpha}}=(\sigma_{\mu})_{\alpha\dot{\alpha}}x^{\mu}, \tag{2.16}$$

where the parameters $a'_{\mu}=\frac{1}{2}(\bar{\sigma}_{\mu})^{\dot{\alpha}\alpha}a'_{\alpha\dot{\alpha}}$ and $w_{\dot{\alpha}}$ satisfy the ADHM constraints

$$(\vec{\tau})^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{w}^{\dot{\beta}}w_{\dot{\alpha}}+\bar{a}'^{\dot{\beta}\alpha}a'_{\alpha\dot{\alpha}})=0,\quad a'_{\mu}=\bar{a}'_{\mu}. \tag{2.17}$$

$\mathcal{M}^I=(\mu^I\ \mathcal{M}'^I_{\alpha})^T$ is the $(N+2k)\times k$ constant Grassmann-odd matrix which satisfies the fermionic ADHM constraints

$$\bar{\mu}^I w_{\dot{\alpha}}+\bar{w}_{\dot{\alpha}}\mu^I+[\mathcal{M}'^{\alpha I},a'_{\alpha\dot{\alpha}}]=0,\quad \mathcal{M}'^I_{\alpha}=\bar{\mathcal{M}}'^I_{\alpha}. \tag{2.18}$$

The parameters $a'_{\alpha\dot{\alpha}}$, $w_{\dot{\alpha}}$, $\mathcal{M}'^I_{\alpha}$ and μ^I are called ADHM moduli. C in (2.15) is the 2×2 matrix of which components are $C_{\alpha}{}^{\beta}=\frac{1}{2}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}C^{\mu\nu}$. The $k\times k$ matrix χ obeys the following equation such that (2.15) is a solution of (2.12):

$$\boldsymbol{L}\chi=i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J+C^{\mu\nu}[a'_{\mu},a'_{\nu}], \tag{2.19}$$

where the operator $\boldsymbol{L}$ is defined by

$$\boldsymbol{L}*=\frac{1}{2}\left\{\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}},*\right\}+\Big[a'_{\mu},[a'^{\mu},*]\Big]. \tag{2.20}$$

In the case of $C^{\mu\nu}=0$, a solution of

$$F^{(+)}_{\mu\nu}=0, \tag{2.21}$$

$$(\sigma^{\mu})_{\alpha\dot{\beta}}D_{\mu}\bar{\Lambda}^{\dot{\beta}}_I=0, \tag{2.22}$$

$$D^2\bar{\varphi}-i\sqrt{2}g\bar{\Lambda}_I\bar{\Lambda}^I=0, \tag{2.23}$$

$$\Lambda^I_{\alpha}=0,\quad \varphi=0 \tag{2.24}$$

satisfies the equations of motion. In this case, the solution is independent of $\bar{C}^{\mu\nu}$ because the self-duality of $\bar{C}^{\mu\nu}$ leads to $\bar{C}^{\mu\nu}F^{(-)}_{\mu\nu}=0$ in (2.8). Therefore the anti-self-dual solution is not deformed by $\bar{C}^{\mu\nu}$ when $C^{\mu\nu}=0$.

Nextly we consider the case where both $C^{\mu\nu}$ and $\bar{C}^{\mu\nu}$ are nonzero and where the adjoint scalar fields $\varphi,\bar{\varphi}$ have nonzero VEVs. In this case, we should consider the constrained

instanton solution (see [34] for a review). We solve the equations of motion perturbatively in the gauge coupling g. The expansion in g gives reliable results when the VEVs $\phi = \langle\varphi\rangle$ and $\bar{\phi} = \langle\bar{\varphi}\rangle$ are large. Then in the self-dual background the classical action S is expanded as

$$S = \frac{8\pi^2 k}{g^2} + ik\theta + g^0 S^{(0)}_{\text{eff}} + \mathcal{O}(g^2). \tag{2.25}$$

$S^{(0)}_{\text{eff}}$ is called the instanton effective action. The instanton effective action in the anti-self-dual background is also defined similarly. $S^{(0)}_{\text{eff}}$ is expressed by the ADHM moduli parameters by plugging the constrained instanton solution into the action.

In the next subsections, we investigate the constrained instanton solutions. We will discuss the solution for the self-dual and the anti-self-dual cases separately.

2.1 Anti-self-dual case

For the anti-self-dual case (2.7), the solution is expanded in the gauge coupling g as

$$A_\mu = g^{-1}A^{(0)}_\mu + gA^{(1)}_\mu + \cdots , \tag{2.26}$$

$$\Lambda^I = g^{\frac{1}{2}}\Lambda^{(0)I} + g^{\frac{5}{2}}\Lambda^{(1)I} + \cdots , \tag{2.27}$$

$$\bar{\Lambda}_I = g^{-\frac{1}{2}}\bar{\Lambda}^{(0)}_I + g^{\frac{3}{2}}\bar{\Lambda}^{(1)}_I + \cdots , \tag{2.28}$$

$$\varphi = g^0\varphi^{(0)} + g^2\varphi^{(1)} + \cdots , \tag{2.29}$$

$$\bar{\varphi} = g^0\bar{\varphi}^{(0)} + g^2\bar{\varphi}^{(1)} + \cdots . \tag{2.30}$$

The equations of motion (2.8) for the fields at the leading order become

$$F^{(0)(+)}_{\mu\nu} = 0, \tag{2.31}$$

$$\nabla^2\bar{\varphi}^{(0)} - i\sqrt{2}\bar{\Lambda}^{(0)}_I\bar{\Lambda}^{(0)I} = 0, \tag{2.32}$$

$$\nabla^2\varphi^{(0)} = 0, \tag{2.33}$$

$$(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu\bar{\Lambda}^{(0)\dot{\beta}}_I = 0, \tag{2.34}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\nabla_\mu\Lambda^{(0)I}_\beta - \sqrt{2}[\varphi^{(0)}, \bar{\Lambda}^{(0)I\dot{\alpha}}] = 0, \tag{2.35}$$

$$\nabla_\mu F^{(0)\mu\nu} = 0, \tag{2.36}$$

where ∇_μ denotes the covariant derivative in the instanton background $\nabla_\mu = \partial_\mu + i[A^{(0)}_\mu, *]$. These equations are not deformed. The instanton effective action $S^{(0)}_{\text{eff}}$ in (2.25) is evaluated

as

$$S_{\rm eff}^{(0)} = \frac{1}{\kappa}\int d^4x\ {\rm Tr}\left[\nabla_\mu\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)} - \frac{i}{\sqrt{2}}\bar{\Lambda}_I^{(0)}[\varphi^{(0)},\bar{\Lambda}^{(0)I}]\right], \tag{2.37}$$

which is not also deformed.

2.2 Self-dual case

For the self-dual case (2.6), we have the expansion

$$A_\mu = g^{-1}A_\mu^{(0)} + gA_\mu^{(1)} + \cdots, \tag{2.38}$$

$$\Lambda^I = g^{-\frac{1}{2}}\Lambda^{(0)I} + g^{\frac{3}{2}}\Lambda^{(1)I} + \cdots, \tag{2.39}$$

$$\bar{\Lambda}_I = g^{\frac{1}{2}}\bar{\Lambda}_I^{(0)} + g^{\frac{5}{2}}\bar{\Lambda}_I^{(1)} + \cdots, \tag{2.40}$$

$$\varphi = g^0\varphi^{(0)} + g^2\varphi^{(1)} + \cdots, \tag{2.41}$$

$$\bar{\varphi} = g^0\bar{\varphi}^{(0)} + g^2\bar{\varphi}^{(1)} + \cdots. \tag{2.42}$$

The equations of motion at the leading order are

$$F_{\mu\nu}^{(0)(-)} = 0, \tag{2.43}$$

$$\nabla^2\bar{\varphi}^{(0)} + iF_{\mu\nu}^{(0)}\bar{C}^{\mu\nu} = 0, \tag{2.44}$$

$$\nabla^2\varphi^{(0)} + i\sqrt{2}\Lambda^{(0)I}\Lambda_I^{(0)} + iF_{\mu\nu}^{(0)}C^{\mu\nu} = 0, \tag{2.45}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\nabla_\mu\Lambda_\beta^{(0)I} = 0, \tag{2.46}$$

$$(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu\bar{\Lambda}_I^{(0)\dot{\beta}} + \sqrt{2}[\bar{\varphi}^{(0)},\Lambda_{I\alpha}^{(0)}] + \sqrt{2}\Lambda^{(0)\beta}{}_I\bar{C}_{(\beta\alpha)} = 0, \tag{2.47}$$

$$\nabla_\mu F^{(0)\mu\nu} = 0, \tag{2.48}$$

The equations (2.48) is automatically satisfied due to the self-dual condition (2.43). Other equations (2.43)–(2.46) have been solved via the ADHM construction in the case of $\bar{C}^{\mu\nu} = 0$ [25]. For nonzero $C^{\mu\nu}$ and $\bar{C}^{\mu\nu}$ these are solved as

$$A_\mu^{(0)} = -i\bar{U}\partial_\mu U, \tag{2.49}$$

$$\Lambda_\alpha^{(0)I} = \bar{U}(\mathcal{M}^I f\bar{b}_\alpha - b_\alpha f\bar{\mathcal{M}}^I)U, \tag{2.50}$$

$$\varphi^{(0)} = -i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{U}\mathcal{M}^I f\bar{\mathcal{M}}^J U + \bar{U}\begin{pmatrix}\phi & 0\\ 0 & \chi\mathbf{1}_2 + \mathbf{1}_k C\end{pmatrix}U, \tag{2.51}$$

$$\bar{\varphi}^{(0)} = \bar{U}\begin{pmatrix}\bar{\phi} & 0\\ 0 & \bar{\chi}\mathbf{1}_2 + \mathbf{1}_k\bar{C}\end{pmatrix}U. \tag{2.52}$$

Here $\bar{C}$ is the 2×2 matrix of which components are $\bar{C}_\alpha{}^\beta = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta \bar{C}^{\mu\nu}$. The $k\times k$ matrices χ and $\bar{\chi}$ satisfy the equations

$$\boldsymbol{L}\chi = i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J + \bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}} + C^{\mu\nu}[a'_\mu, a'_\nu], \tag{2.53}$$

$$\boldsymbol{L}\bar{\chi} = \bar{w}^{\dot{\alpha}}\bar{\phi} w_{\dot{\alpha}} + \bar{C}^{\mu\nu}[a'_\mu, a'_\nu]. \tag{2.54}$$

We note that we do not need to solve the equation of motion for $\bar{\Lambda}_I^{(0)\dot{\alpha}}$ explicitly. This is because contribution of $\bar{\Lambda}_I^{(0)\dot{\alpha}}$ to the action is just the subleading order in gauge coupling constant g. We also note that the solutions of the gauge field A_μ and Weyl fermion $\Lambda_\alpha^{(0)I}$ are not deformed by $C^{\mu\nu}$ and $\bar{C}^{\mu\nu}$ and the ADHM constraints (2.17) and (2.18) (see appendix A) do not change. This is contrasted with the case of $\mathcal{N}=1$ non(anti)commutative deformed super Yang-Mills [15, 14] in which the bosonic ADHM constraints (2.17) are changed due to the non-zero graviphoton background while fermionic constraints (2.18) remain unchanged.

Now let us evaluate the instanton effective action in the self-dual instanton background and write down it in terms of the ADHM moduli. Some formulae are proved in appendix B. By substituting the expansion (2.38)-(2.42) into the classical action, the instanton effective action is given by

$$\begin{aligned} S_{\text{eff}}^{(0)} = \frac{1}{\kappa}\int d^4x\ \text{Tr}\Big[\nabla_\mu\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)} - \frac{i}{\sqrt{2}}\Lambda^{(0)I}[\bar{\varphi}, \Lambda_I^{(0)}] - i\bar{\varphi}^{(0)}F_{\mu\nu}^{(0)}C^{\mu\nu} \\ -i\varphi^{(0)}F_{\mu\nu}^{(0)}\bar{C}^{\mu\nu} - \frac{i}{\sqrt{2}}\Lambda_\alpha^{(0)I}\Lambda_{\beta I}^{(0)}\bar{C}^{(\alpha\beta)}\Big]. \end{aligned} \tag{2.55}$$

From the equation of motion (2.44), the first and the fourth terms in (2.55) become the total derivative,

$$\frac{1}{\kappa}\int d^4x\ \text{Tr}\left[\nabla_\mu\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)} - i\varphi^{(0)}F_{\mu\nu}^{(0)}\bar{C}^{\mu\nu}\right] = \int d^4x\ \frac{1}{\kappa}\text{Tr}\Big[\partial_\mu\big(\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)}\big)\Big], \tag{2.56}$$

which is evaluated by the value of $\nabla_\mu\varphi^{(0)}$ at infinity as

$$\begin{aligned} \int d^4x\ \frac{1}{\kappa}\text{Tr}\Big[\partial_\mu\big(\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)}\big)\Big] &= \frac{1}{\kappa}\lim_{|x|\to\infty} 2\pi^2|x|^2x^\mu\text{Tr}\big[\varphi^{(0)}\nabla_\mu\bar{\varphi}^{(0)}\big] \\ &= \frac{4\pi^2}{\kappa}\ \text{tr}_k\left[\frac{1}{2}\bar{w}^{\dot{\alpha}}(\bar{\phi}\phi + \phi\bar{\phi})w_{\dot{\alpha}} - \bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}}\bar{\chi}\right]. \end{aligned} \tag{2.57}$$

Here tr_k denotes the trace for instanton indices. The second and the fifth terms in (2.55) are calculated as (see appendix B)

$$\begin{aligned}\int d^4x\,\frac{1}{\kappa}\mathrm{Tr}\Big[-\frac{i}{\sqrt{2}}\Lambda^{(0)\alpha I}\big[\bar{\varphi}^{(0)},\Lambda^{(0)}_{\alpha I}\big]-\frac{i}{\sqrt{2}}\bar{C}^{(\alpha\beta)}\Lambda^{(0)I}_{\alpha}\Lambda^{(0)}_{\beta I}\Big]\\ =\frac{1}{\kappa}\sqrt{2}\pi^2 i\epsilon_{IJ}\mathrm{tr}_k\Big[\bar{\mu}^I\bar{\phi}\mu^J-\bar{\mathcal{M}}^I\mathcal{M}^J\bar{\chi}+\frac{1}{2}\bar{C}^{(\alpha\beta)}\mathcal{M}'^I_{\alpha}\mathcal{M}'^J_{\beta}\Big].\end{aligned}\tag{2.58}$$

The third term is

$$\int d^4x\,\frac{1}{\kappa}\mathrm{Tr}\Big[-i\bar{\varphi}^{(0)}F^{(0)}_{\mu\nu}C^{\mu\nu}\Big]=\frac{\pi^2}{\kappa}\mathrm{tr}_k\Big[-4C^{\mu\nu}[a'_\mu,a'_\nu]\bar{\chi}+C^{\mu\nu}\bar{C}_{\mu\nu}\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}\Big].\tag{2.59}$$

Finally $S^{(0)}_{\mathrm{eff}}$ becomes

$$\begin{aligned}S^{(0)}_{\mathrm{eff}}=\frac{4\pi^2}{\kappa}\mathrm{tr}_k\Big[&-\Big(\bar{w}^{\dot{\alpha}}\bar{\phi}w_{\dot{\alpha}}+\bar{C}^{\mu\nu}[a'_\mu,a'_\nu]\Big)\boldsymbol{L}^{-1}\Big(i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J+\bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}}+C^{\mu\nu}[a'_\mu,a'_\nu]\Big)\\ &+i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mu}^I\bar{\phi}\mu^J+\frac{1}{2}\bar{w}^{\dot{\alpha}}(\bar{\phi}\phi+\phi\bar{\phi})w_{\dot{\alpha}}-i\frac{\sqrt{2}}{8}\bar{C}^{(\alpha\beta)}\epsilon_{IJ}\mathcal{M}'^I_{\alpha}\mathcal{M}'^J_{\beta}\\ &+\frac{1}{4}C^{\mu\nu}\bar{C}_{\mu\nu}\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}\Big].\end{aligned}\tag{2.60}$$

This action can be also obtained from the following action by integrating over the auxiliary fields χ, $\bar{\chi}$, $\bar{\psi}^{\dot{\alpha}}_I$ and $\vec{D}$

$$\begin{aligned}S^{(0)}_{\mathrm{eff}}=\frac{2\pi^2}{\kappa}\mathrm{tr}_k\Big[&-2\Big([\bar{\chi},a'_\mu]-\bar{C}_{\mu\nu}a'^\nu\Big)\Big([\chi,a'^\mu]-C^{\mu\rho}a'_\rho\Big)\\ &+(\bar{\chi}\bar{w}^{\dot{\alpha}}-\bar{w}^{\dot{\alpha}}\bar{\phi})(w_{\dot{\alpha}}\chi-\phi w_{\dot{\alpha}})+(\chi\bar{w}^{\dot{\alpha}}-\bar{w}^{\dot{\alpha}}\phi)(w_{\dot{\alpha}}\bar{\chi}-\bar{\phi}w_{\dot{\alpha}})\\ &-i\frac{\sqrt{2}}{2}\bar{\mu}^I\epsilon_{IJ}(\mu^J\bar{\chi}-\bar{\phi}\mu^J)-i\frac{\sqrt{2}}{4}\mathcal{M}'^{\alpha I}\epsilon_{IJ}\Big([\bar{\chi},\mathcal{M}'^J_{\alpha}]-\bar{C}_{(\alpha\beta)}\mathcal{M}'^{\beta J}\Big)\\ &+\frac{1}{2}C^{\mu\nu}\bar{C}_{\mu\rho}(\delta_\nu{}^\rho\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}+4a'_\nu a'^\rho)\Big]+S_{\mathrm{ADHM}},\end{aligned}\tag{2.61}$$

where S_{ADHM} contains the Lagrange multipliers $\bar{\psi}^{\dot{\alpha}}_I,\vec{D}$ associated with the ADHM constraints (2.17) and (2.18) by its equation of motion. It is given by

$$\begin{aligned}S_{\mathrm{ADHM}}=\frac{4\pi^2}{\kappa}\mathrm{tr}_k\Big[&-i\bar{\psi}^{\dot{\alpha}}_I\Big(\bar{\mu}^I w_{\dot{\alpha}}+\bar{w}_{\dot{\alpha}}\mu^I+[\mathcal{M}'^{\alpha I},a'_{\alpha\dot{\alpha}}]\Big)\\ &-i\vec{D}\cdot\vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}}\Big(\bar{w}^{\dot{\beta}}w_{\dot{\alpha}}+\bar{a}'^{\dot{\beta}\alpha}a'_{\alpha\dot{\alpha}}\Big)\Big].\end{aligned}\tag{2.62}$$

We note that this effective action (2.61) is different from the fractional D3/D(-1) action in the R-R 3-form background at $\mathcal{O}(C\bar{C})$ which is obtained as [25]

$$\begin{aligned}
S^{(0)}_{\mathrm{str}} = \frac{2\pi^2}{\kappa}\mathrm{tr}_k\Big[&-2\Big([\bar{\chi}, a'_\mu] - \bar{C}_{\mu\nu}a'^{\nu}\Big)\Big([\chi, a'^{\mu}] - C^{\mu\rho}a'_\rho\Big)\\
&+ (\bar{\chi}\bar{w}^{\dot{\alpha}} - \bar{w}^{\dot{\alpha}}\bar{\phi})(w_{\dot{\alpha}}\chi - \phi w_{\dot{\alpha}}) + (\chi\bar{w}^{\dot{\alpha}} - \bar{w}^{\dot{\alpha}}\phi)(w_{\dot{\alpha}}\bar{\chi} - \bar{\phi}w_{\dot{\alpha}})\\
&- i\frac{\sqrt{2}}{2}\bar{\mu}^I\epsilon_{IJ}(\mu^J\bar{\chi} - \bar{\phi}\mu^J) - i\frac{\sqrt{2}}{4}\mathcal{M}'^{\alpha I}\epsilon_{IJ}\Big([\bar{\chi}, \mathcal{M}'^J_\alpha] - \bar{C}_{(\alpha\beta)}\mathcal{M}'^{\beta J}\Big)\Big]\\
&+ S_{\mathrm{ADHM}}. \qquad (2.63)
\end{aligned}$$

The difference between (2.63) and (2.61) is

$$\begin{aligned}
S^{(0)}_{\mathrm{str}} - S^{(0)}_{\mathrm{eff}} &= -\frac{\pi^2}{\kappa}\mathrm{tr}_k\Big[C^{\mu\nu}\bar{C}_{\mu\rho}(\delta_\nu{}^\rho\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}} + 4a'_\nu a'^\rho)\Big]\\
&= -\frac{\pi^2}{\kappa}\mathrm{tr}_k\Big[C^{\mu\nu}\bar{C}_{\mu\nu}(\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}} + a'_\rho a'^\rho)\Big]. \qquad (2.64)
\end{aligned}$$

Here we have used the relation from the self-duality of $C^{\mu\nu}$ and $\bar{C}^{\mu\nu}$

$$C_{\mu\rho}\bar{C}_\nu{}^\rho + C_{\nu\rho}\bar{C}_\mu{}^\rho = \frac{1}{2}C_{\rho\sigma}\bar{C}^{\rho\sigma}\delta_{\mu\nu}. \qquad (2.65)$$

In order to recover the effective action of the D3/D(-1)-branes (2.63) from the (S,A)-deformed super Yang-Mills at $\mathcal{O}(C\bar{C})$, we find that the term

$$\delta\mathcal{L} = -\frac{g^2}{16\kappa}C^{\rho\sigma}\bar{C}_{\rho\sigma}|x|^2\mathrm{Tr}\,[F^{\mu\nu}F_{\mu\nu}] \qquad (2.66)$$

needs to be added to the space-time Lagrangian (2.2). The contribution $\delta S^{(0)}_{\mathrm{eff}}$ to the instanton effective action coming from (2.66) is evaluated and coincides with (2.64) (see appendix B for detail). Then $S^{(0)}_{\mathrm{eff}} + \delta S^{(0)}_{\mathrm{eff}}$ completely agrees with $S^{(0)}_{\mathrm{str}}$. We note that the term (2.66) which contains space-time coordinates explicitly cannot be calculated in our previous paper [31] in which we have treated the constant R-R background only. We also note that the term (2.66) does not change the self-dual equation at the leading order (2.43)–(2.46). Hence when we start from the improved space-time Lagrangian $\mathcal{L}+\delta\mathcal{L}$, we find the same self-dual solution (2.50)–(2.52) and obtain (2.63) as the instanton effective action of the improved theory.

The evaluation of the instanton effective action in string theory is based on the D(-1)-brane effective action. In the presence of the R-R background, the modulus a'_μ is stabilized

at the origin due to the $\mathcal{O}(C\bar{C})$ contribution in (2.63) which is regarded as the mass term of a'_μ. This moduli stabilization breaks translational invariance in the D3-brane world-volume. However, from the viewpoint of the D3-brane effective action, namely (S,A)-deformed super Yang-Mills theory, the background does not induce any terms violating translational symmetry . This is the reason why there is no a'_μ mass term $\bar{C}^{\mu\nu}C_{\nu\rho}a'_\mu a'^\rho$ in the field theory calculation in (2.63).

The action (2.63) is invariant under the following deformed supersymmetry transformation

$$\begin{aligned}
\delta a'_{\alpha\dot{\alpha}} &= i\bar{\xi}_{\dot{\alpha}I}\mathcal{M}'^{\ I}_{\alpha}, & \delta\mathcal{M}'^{\ I}_{\alpha} &= -2\sqrt{2}\epsilon^{IJ}\bar{\xi}^{\dot{\alpha}}{}_J[a'_{\alpha\dot{\alpha}},\chi] + 2\sqrt{2}\bar{\xi}^{\dot{\alpha}I}(\sigma^\mu)_{\alpha\dot{\alpha}}C^{\mu\nu}a'_\nu, \\
\delta w_{\dot{\alpha}} &= i\bar{\xi}_{\dot{\alpha}I}\mu^I, & \delta\mu^I &= -2\sqrt{2}\epsilon^{IJ}\bar{\xi}^{\dot{\alpha}}{}_J(w_{\dot{\alpha}}\chi - \phi w_{\dot{\alpha}}), \\
\delta\chi &= 0, & \delta\bar{\chi} &= -\sqrt{2}i\epsilon^{IJ}\bar{\xi}_{\dot{\alpha}I}\bar{\psi}^{\dot{\alpha}}{}_J, \\
\delta\vec{D} &= -\sqrt{2}\vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\xi}_{\dot{\alpha}}{}^I[\bar{\psi}^{\dot{\beta}}{}_I,\chi], & \delta\bar{\psi}^{\dot{\alpha}}{}_I &= 2[\chi,\bar{\chi}]\bar{\xi}^{\dot{\alpha}}{}_I - i\vec{D}\cdot\vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\xi}^{\dot{\beta}}{}_I,
\end{aligned} \tag{2.67}$$

when $C^{\mu\rho}\bar{C}_{\rho\nu} = \bar{C}^{\mu\rho}C_{\rho\nu}$. As we will see in next section, this condition is equivalent to the flatness of the Ω-background. After the topological twist, the above symmetry becomes the BRST symmetry which is important to apply the localization technique [27, 26, 28, 29, 30] for the calculation of the prepotential. The instanton effective action is BRST-exact as shown in [25].

3 Relation to the Ω-background deformation

In the previous section, we showed that the instanton effective action in the (S,A)-deformed $\mathcal{N}=2$ super Yang-Mills theory coincides with the D3/D(−1)-brane effective action with (S,A)-type background if we introduce the additional term (2.66). In the following, we discuss the relation between the Ω-background deformation and the (S,A)-deformation of the $\mathcal{N}=2$ super Yang-Mills theory.

The four-dimensional Ω-deformed $\mathcal{N}=2$ super Yang-Mills Lagrangian $\mathcal{L}(\Omega,\bar{\Omega})$ is obtained by the dimensional reduction of six-dimensional $\mathcal{N}=1$ super Yang-Mills theory in the Ω-background metric [35]

$$ds_6^2 = 2dzd\bar{z} + (dx^\mu + \bar{\Omega}^\mu dz + \Omega^\mu d\bar{z})^2, \tag{3.1}$$

where $z = \frac{1}{\sqrt{2}}(x^5 - ix^6)$, $\bar{z} = \frac{1}{\sqrt{2}}(x^5 + ix^6)$. Ω^μ and $\bar{\Omega}^\mu$ are defined by $\Omega^\mu \equiv \Omega^{\mu\nu}x_\nu$, $\bar{\Omega}^\mu \equiv \bar{\Omega}^{\mu\nu}x_\nu$ with constant anti-symmetric matrices $\Omega^{\mu\nu} = -\Omega^{\nu\mu}$ and $\bar{\Omega}^{\mu\nu} = -\bar{\Omega}^{\nu\mu}$. In this background, all nonzero components in the Riemann tensor are proportional to $\Omega_{\mu\nu}\bar{\Omega}^\nu{}_\rho - \bar{\Omega}_{\mu\nu}\Omega^\nu{}_\rho$. Then Ω and $\bar{\Omega}$ are taken to be commutative matrices so that space-time is flat. As we will see, under the identification (3.11), this flatness condition becomes the supersymmetry invariance of the instanton effective action.

The six dimensional $\mathcal{N} = 1$ super Yang-Mills action is

$$S = \int d^6x \, \sqrt{-g}\mathrm{Tr}\left[-\frac{1}{4}g^{MP}g^{NQ}F_{MN}F_{PQ} - \frac{i}{2}\bar{\Psi}e^M{}_m\Gamma^m\mathcal{D}_M\Psi \right], \tag{3.2}$$

where $M, N = 0, \cdots 5$ stands for curved indices in six dimensional space-time and m is a local Lorentz index. $e^M{}_m$ is a vielbein and Γ^m is a six dimensional gamma matrix. The covariant derivative is defined by $\mathcal{D}_M = D_M - \frac{1}{2}\omega_{M,mn}\Gamma^{mn}$ where D_M is an ordinary gauge covariant derivative and $\omega_{M,mn}$ is a spin connection. The field strength is defined by $F_{MN} = \partial_M A_N - \partial_N A_M + ig[A_M, A_N]$ and Ψ is a six dimensional Dirac spinor. After the dimensional reduction and the Wick rotation, we obtain the four-dimensional Lagrangian

$$\mathcal{L}(\Omega, \bar{\Omega}) = \mathcal{L}_0 + \delta\mathcal{L}(\Omega, \bar{\Omega}), \tag{3.3}$$

where $\mathcal{L}_0$ is the $\mathcal{N} = 2$ super Yang-Mills Lagrangian (2.1) and $\delta\mathcal{L}(\Omega, \bar{\Omega})$ is

$$\begin{aligned}
\delta\mathcal{L}(\Omega, \bar{\Omega}) \;=\; & \frac{1}{k}\mathrm{Tr}\Big[\, gF_{\mu\nu}D^\mu\bar{\varphi}\Omega^\nu + gF_{\mu\nu}D^\mu\varphi\bar{\Omega}^\nu \\
& +ig^2 D_\mu\bar{\varphi}[\varphi, \bar{\varphi}]\Omega^\mu + ig^2 D_\mu\varphi[\varphi, \bar{\varphi}]\bar{\Omega}^\mu - g^2 F_{\mu\rho}F_\nu{}^\rho\Omega^\mu\bar{\Omega}^\nu + \frac{g^2}{2}D_\mu\bar{\varphi}D_\nu\bar{\varphi}\Omega^\mu\Omega^\nu \\
& -g^2 D_\mu\bar{\varphi}D_\nu\varphi\Omega^\mu\bar{\Omega}^\nu + \frac{g^2}{2}D_\mu\varphi D_\nu\varphi\bar{\Omega}^\mu\bar{\Omega}^\nu + ig^3[\varphi, \bar{\varphi}]F_{\mu\nu}\bar{\Omega}^\mu\Omega^\nu \\
& -\frac{g}{\sqrt{2}}\Lambda^{\alpha I}D_\mu\Lambda_{\alpha I}\bar{\Omega}^\mu - \frac{g}{\sqrt{2}}\bar{\Lambda}_{\dot{\alpha}I}D_\mu\bar{\Lambda}^{\dot{\alpha}I}\Omega^\mu \\
& -\frac{g}{\sqrt{2}}\Lambda_\alpha{}^I\Lambda_{\beta I}\bar{\Omega}^{(\alpha\beta)} - \frac{g}{\sqrt{2}}\bar{\Lambda}_{\dot{\alpha}I}\bar{\Lambda}_{\dot{\beta}}{}^I\Omega^{(\dot{\alpha}\dot{\beta})}\Big] + \mathcal{O}(\Omega^3, \bar{\Omega}^3).
\end{aligned} \tag{3.4}$$

Here $\bar{\Omega}^{(\alpha\beta)} = \frac{1}{2}\varepsilon^{\alpha\gamma}(\sigma_{\mu\nu})_\gamma{}^\beta\Omega^{\mu\nu}$, $\Omega^{(\dot{\alpha}\dot{\beta})} = \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\gamma}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\gamma}}\bar{\Omega}^{\mu\nu}$, $\bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$. This part $\delta\mathcal{L}(\Omega, \bar{\Omega})$ can be interpreted as a shift of the scalar fields $(\varphi, \bar{\varphi}) \to (\varphi - i\Omega^\mu D_\mu, \bar{\varphi} + i\bar{\Omega}^\mu D_\mu)$ and the modification of the complex coupling constant $\tau \to \tau - \frac{4\sqrt{2}i}{g^2}\theta_\alpha\bar{\theta}_\beta\bar{\Omega}^{(\alpha\beta)}$ in the $\mathcal{N} = 2$ superfield formalism of super Yang-Mills action [29].

Notice that once we assume the self-duality condition of $\bar{\Omega}^{\mu\nu}$, the term which contains $\bar{\Omega}^{(\dot{\alpha}\dot{\beta})}$ vanish due to the anti-self-duality of $\bar{\sigma}_{\mu\nu}$. In the following, we assume self-duality of $\Omega^{\mu\nu}$ and $\bar{\Omega}^{\mu\nu}$.

The Lagrangian (3.3) contains many x^μ-dependent interactions and looks like quite different from the (S,A)-deformed theory. However, the leading order equations of motion for the self-dual instanton (2.38)-(2.42) turn out to be not so different from the (S,A)-deformed theory. It is given by

$$\nabla^2\varphi^{(0)} - (\nabla^\mu F^{(0)}_{\mu\nu})\Omega^\nu + F^{(0)}_{\mu\nu}\Omega^{\mu\nu} + \sqrt{2}i\Lambda^{(0)\alpha I}\Lambda^{(0)}_{\alpha I} = 0, \tag{3.5}$$

$$\nabla^2\bar{\varphi}^{(0)} - (\nabla^\mu F^{(0)}_{\mu\nu})\bar{\Omega}^\nu + F^{(0)}_{\mu\nu}\bar{\Omega}^{\mu\nu} = 0, \tag{3.6}$$

$$\nabla^\mu(F^{(0)}_{\mu\nu} + \tilde{F}^{(0)}_{\mu\nu}) = 0, \tag{3.7}$$

$$F^{(0)}_{\mu\nu} = \tilde{F}^{(0)}_{\mu\nu}, \tag{3.8}$$

$$i(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu\bar{\Lambda}^{(0)\dot{\beta}}{}_I + \sqrt{2}i[\bar{\varphi}^{(0)}, \Lambda^{(0)}_{\alpha I}] - \sqrt{2}\Lambda^{(0)}_{\beta I}\bar{\Omega}_\alpha{}^\beta + \frac{1}{\sqrt{2}}\nabla_\mu\Lambda^{(0)}_{\alpha I}\bar{\Omega}^\mu = 0, \tag{3.9}$$

$$i(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\nabla_\mu\Lambda^{(0)I}_\alpha = 0. \tag{3.10}$$

The Bianchi identity and the self-dual condition $F^{(0)}_{\mu\nu} = \tilde{F}^{(0)}_{\mu\nu}$ can be used to remove the second term in (3.5) and (3.6). After the identification

$$\Omega_{\mu\nu} = iC_{\mu\nu},\ \bar{\Omega}_{\mu\nu} = i\bar{C}_{\mu\nu}, \tag{3.11}$$

the leading order equations of motion (3.5)–(3.10) agree with the equations (2.43)-(2.48) for (S,A)-deformed super Yang-Mills theory except for the equation of motion for $\bar{\Lambda}$. However the contribution of $\bar{\Lambda}^{(0)}_I$ is just the subleading order in g, hence it does not contribute to the instanton effective action. The $\mathcal{O}(g^0)$ terms in the instanton effective action (3.4) is given by

$$\begin{aligned} S^{(0)}_{\rm eff}(\Omega,\bar{\Omega}) &= \frac{1}{\kappa}\int d^4x\ \mathrm{Tr}\Big[\nabla_\mu\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)} - \frac{i}{\sqrt{2}}\Lambda^{(0)I}[\bar{\varphi}, \Lambda^{(0)}_I] - \bar{\varphi}^{(0)}F^{(0)}_{\mu\nu}\Omega^{\mu\nu} \\ &\qquad -\varphi^{(0)}F^{(0)}_{\mu\nu}\bar{\Omega}^{\mu\nu} + \frac{1}{\sqrt{2}}\Lambda^{(0)I}_\alpha\Lambda^{(0)}_{\beta I}\bar{\Omega}^{(\alpha\beta)}\Big] \\ &\quad +\frac{1}{\kappa}\int d^4x\ \mathrm{Tr}\Big[\ F^{(0)}_{\mu\rho}F^{(0)\rho}_\nu\Omega^\mu\bar{\Omega}^\nu + \frac{1}{\sqrt{2}}\Lambda^{(0)\alpha I}\nabla_\mu\Lambda^{(0)}_{\alpha I}\bar{\Omega}^\mu\Big]. \end{aligned} \tag{3.12}$$

The last term in (3.12) can be rewritten as

$$\frac{1}{\sqrt{2}}\mathrm{Tr}\Big[\bar{\Omega}^\mu\Lambda^{(0)\alpha I}\nabla_\mu\Lambda^{(0)}_{\alpha I}\Big] = \frac{1}{\sqrt{2}}\bar{\Omega}^{\mu\nu}\mathrm{Tr}\Big[\Lambda^{(0)I}\sigma_{\mu\nu}\Lambda^{(0)}_I - \partial^\rho\big(x_\nu\Lambda^{(0)I}\sigma_{\mu\rho}\Lambda^{(0)}_I\big)\Big]. \tag{3.13}$$

Here the second term in the right hand side is the total derivative and does not contribute to the effective action in the instanton background since $x_\nu \Lambda^{(0)I}\sigma_{\mu\rho}\Lambda^{(0)}_I$ behaves as $|x|^{-5}$ for large $|x|$. Therefore we have

$$\begin{aligned} S^{(0)}_{\text{eff}}(\Omega,\bar{\Omega}) &= \frac{1}{\kappa}\int d^4x \ \mathrm{Tr}\left[\nabla_\mu\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)} - \frac{i}{\sqrt{2}}\Lambda^{(0)I}[\bar{\varphi},\Lambda^{(0)}_I] - \bar{\varphi}^{(0)}F^{(0)}_{\mu\nu}\Omega^{\mu\nu}\right. \\ &\qquad \left. -\varphi^{(0)}F^{(0)}_{\mu\nu}\bar{\Omega}^{\mu\nu} - \frac{1}{\sqrt{2}}\Lambda^{(0)I}_\alpha\Lambda^{(0)}_{\beta I}\bar{\Omega}^{(\alpha\beta)}\right] \\ &\quad +\frac{1}{\kappa}\int d^4x \ \mathrm{Tr}\left[F^{(0)}_{\mu\rho}F^{(0)\rho}_{\nu}\Omega^\mu\bar{\Omega}^\nu\right]. \end{aligned} \tag{3.14}$$

This result coincides with the one obtained from the improved action discussed in section 2. The last term in (3.14) agrees with (2.66) by using the relation (2.65) and self-duality of $F^{(0)}_{\mu\nu}$.

We note that we compared the space-time action deformed in the R-R 3-form background with the action in the Ω-background without the R-symmetry gauge field Wilson line. If one includes the R-symmetry gauge field Wilson line, one gets the topological field theory in the Ω-background [27, 26] which differs from the $\mathcal{N}=2$ action in the same background by topological terms [29]. Therefore the instanton effective action remains the same by the twisting. The deformed action is BRST-exact and the instanton effective action is also written in the BRST-exact form. Although it is not clear at this moment how to introduce the R-symmetry gauge Wilson line in the fractional D3-branes, the BRST transformations would correspond to the deformed supersymmetry transformation in the R-R 3-form background.

4 Conclusions and discussions

In this paper, we investigate (anti-)self-dual solutions in the deformed $\mathcal{N}=2$ super Yang-Mills theory. The theory is realized on the (fractional) D3-branes at the fixed point of the orbifold $\mathbf{C}\times\mathbf{C}^2/\mathbf{Z}_2$ in the presence of the R-R 3-form field strength background. The R-R 3-form background $\mathcal{F}^{(\alpha\beta)[AB]}$ is scaled as $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}=$ fixed in order to give the deformation parameters $C,\bar{C}$ the mass dimension one. In the $\mathcal{N}=2$ supergravity context, these are interpreted as the graviphoton and the vector backgrounds, respectively [25].

The instanton solution is expressed in terms of the the ADHM moduli and the deformation parameters. With this solution, we explicitly evaluate the instanton effective action for the self-dual solution using the field theoretical method. The result agrees with the one previously obtained in the string theory calculation [25] up to the first order in the deformation parameters but differs from that at $\mathcal{O}(C\bar{C})$. However, once we add the translational symmetry breaking term to the (S,A)-deformed action and consider the improved action, we obtain the string theory result.

The deformed $\mathcal{N}=2$ instanton effective action derived from the improved action is the same with the the action in Ω-background [27] despite the fact that the space-time action has a different form. The instanton effective action is invariant under deformed supersymmetry if C and $\bar{C}$ commute with each other, which corresponds to the flatness condition of the Ω-background.

It is interesting to consider the deformation in the (A,S)-type background. In [31], we have shown that the (A,S)-type R-R 3-form background $\mathcal{F}^{[\alpha\beta](AB)}$ induces mass terms for the chiral fermion Λ and other adjoint scalar field interactions[2]. In self-dual case, we can show that the bosonic interactions induced by the (A,S)-background are sub-leading order and do not contribute to the instanton effective action. The only relevant part is the mass term for Λ which contributes to the equation of motion of $\bar{\Lambda}$. However, as we have seen section 2, the solution of $\bar{\Lambda}$ does not contribute to the instanton effective action because it enters in the space-time action as the sub-leading part. Therefore the only modification in the instanton effective action by the (A,S)-background is just the mass term of the Λ which can be easily evaluated by Corrigan's inner product formula (B.7). Similar to the (S,A)-type deformation, there are no (A,S)-background corrections to the instanton effective action for the anti-self-dual case because corrections are sub-leading order.

It is possible to generalize the results in this paper to $\mathcal{N}=4$ and $\mathcal{N}=2^*$ super Yang-Mills theories. These generalizations will appear in a forthcoming paper [32].

[2] In [31], the mass terms for the anti-chiral fermion $\bar{\Lambda}$ was considered. Here we consider different chirality of the (A,S)-type background considered in [31] to generate the mass term for Λ.

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A The ADHM construction in (deformed) $\mathcal{N}=2$ supersymmetric Yang-Mills Theory

Here we briefly summarize the ADHM construction [5, 34]. As we have seen in the equations of motion (2.43), (2.46) for the (S,A)-deformed action, the self-dual equations for the gauge field and the spinor field do not change in the deformed theory. Therefore one can solve them by the ADHM construction based on the undeformed theory. We introduce the $(N+2k)\times 2k$ matrix $\Delta_{\lambda j\dot{\alpha}}$ which is given by

$$\Delta_{\lambda j\dot{\alpha}} = a_{\lambda j\dot{\alpha}} + b_{\lambda j}{}^{\beta}\sigma_{\mu\beta\dot{\alpha}}x^{\mu}, \tag{A.1}$$

where $\lambda = 1,2,\ldots,N+2k$ and $i,j=1,2,\ldots,k$. k is the instanton number. $a_{\lambda j\dot{\alpha}}$ and $b_{\lambda j}{}^{\beta}$ are the constant matrices. They are decomposed as

$$a_{\lambda j\dot{\alpha}} = \begin{pmatrix} w_{uj\dot{\alpha}} \\ (a'_{ij})_{\alpha\dot{\alpha}} \end{pmatrix}, \quad b_{\lambda j}{}^{\beta} = \begin{pmatrix} 0 \\ \delta_{ij}\delta_{\alpha}{}^{\beta} \end{pmatrix}, \quad \lambda = u + i\alpha, \quad u = 1,2,\ldots,N. \tag{A.2}$$

The matrix Δ should satisfy the following ADHM constraints,

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}\Delta_{\lambda j\dot{\beta}} = (f^{-1})_{ij}\delta^{\dot{\alpha}}{}_{\dot{\beta}}, \quad f = \left[\frac{1}{2}\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}} + (x_\mu + a'_\mu)^2\right]^{-1}, \quad a'_\mu = \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}a'_{\alpha\dot{\alpha}}, \tag{A.3}$$

where a'_μ, $w_{\dot{\alpha}}$, and $\bar{w}^{\dot{\alpha}}$ are called ADHM moduli. The ADHM constraints in terms of $a'_{\alpha\dot{\alpha}}, \bar{w}_{\dot{\alpha}}$ are

$$(\vec{\tau})^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{w}^{\dot{\beta}}w_{\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha}a'_{\alpha\dot{\alpha}}) = 0, \quad a'_\mu = \bar{a}'_\mu. \tag{A.4}$$

We also introduce $(N+2k)\times N$ matrix U which satisfies

$$\bar{\Delta}U=0,\quad \bar{U}U=\mathbf{1}_N,\quad U\bar{U}+\Delta_{\dot{\alpha}}f\bar{\Delta}^{\dot{\alpha}}=\mathbf{1}_{N+2k}, \tag{A.5}$$

where $\mathbf{1}_n$ is the $n\times n$ identity matrix. The self-dual gauge field is constructed from U as

$$A_\mu^{(0)}=-i\bar{U}\partial_\mu U. \tag{A.6}$$

The corresponding field strength $F_{\mu\nu}^{(0)}$ is

$$F_{\mu\nu}^{(0)}=-4i\bar{U}b^\alpha(\sigma_{\mu\nu})_\alpha{}^\beta f\bar{b}_\beta U. \tag{A.7}$$

The self-duality of $F_{\mu\nu}^{(0)}$ immediately follows from that of $\sigma_{\mu\nu}$.

Nextly we consider the fermionic moduli which appear as the fermionic zero modes on the instanton background. We solve the Dirac equation on the self-dual background $\bar{\sigma}^\mu\nabla_\mu\Lambda^{(0)I}=0$. The ansatz of the solution is

$$\Lambda_\alpha^{(0)I}=\Lambda_\alpha(\mathcal{M}^I)=\bar{U}(\mathcal{M}^I f\bar{b}_\alpha-b_\alpha f\bar{\mathcal{M}}^I)U, \tag{A.8}$$

where $\mathcal{M}^I$ is the $(N+2k)\times k$ constant matrix. Plugging (A.8) to the Dirac equation, we obtain

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\nabla_\mu\Lambda_\alpha^{(0)I}=2\bar{U}b^\alpha f(\bar{\mathcal{M}}^I\Delta^{\dot{\alpha}}+\bar{\Delta}^{\dot{\alpha}}\mathcal{M}^I)f\bar{b}_\alpha U. \tag{A.9}$$

Then we have the fermionic ADHM constraint

$$\bar{\mathcal{M}}^I\Delta^{\dot{\alpha}}+\bar{\Delta}^{\dot{\alpha}}\mathcal{M}^I=0, \tag{A.10}$$

or equivalently

$$\bar{\mu}^I w_{\dot{\alpha}}+\bar{w}_{\dot{\alpha}}\mu^I+[\mathcal{M}'^{\alpha I},a'_{\alpha\dot{\alpha}}]=0,\quad \mathcal{M}'^I_\alpha=\bar{\mathcal{M}}'^I_\alpha, \tag{A.11}$$

where we have decomposed $\mathcal{M}^I$ as

$$\mathcal{M}^I_{\lambda j}=\begin{pmatrix}\mu^I_{uj}\\ (\mathcal{M}'^I_\alpha)_{ij}\end{pmatrix}. \tag{A.12}$$

$\mathcal{M}'^I_\alpha$, μ^I, $\bar{\mu}^I$ are called fermionic ADHM moduli.

Now we solve the equation of motion of the scalar field $\varphi^{(0)}$

$$\nabla^2\varphi^{(0)}+i\sqrt{2}\Lambda^{(0)I}\Lambda_I^{(0)}+iC^{\mu\nu}F_{\mu\nu}^{(0)}=0. \tag{A.13}$$

First we consider the case of $C^{\mu\nu}=0$. The ansatz of the solution [34] is

$$\varphi^{(0)}=-i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{U}\mathcal{M}^I f\bar{\mathcal{M}}^J U+\bar{U}\begin{pmatrix}\phi & 0\\ 0 & \chi\mathbf{1}_2\end{pmatrix}U. \tag{A.14}$$

The asymptotic behavior of (A.14) is given by $\lim_{|x|\to\infty}\varphi^{(0)}=\phi$. Computing $\nabla^2\varphi^{(0)}$, one can show that

$$\begin{aligned}\nabla^2\varphi^{(0)}&=-i\sqrt{2}\epsilon_{IJ}\Lambda^{(0)I}\Lambda^{(0)J}+4\bar{U}bf\left[i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J-\{f^{-1},\chi\}+\bar{\Delta}^{\dot{\alpha}}\begin{pmatrix}\phi & 0\\ 0 & \chi\mathbf{1}_2\end{pmatrix}\Delta_{\dot{\alpha}}\right]f\bar{b}U\\ &=-i\sqrt{2}\epsilon_{IJ}\Lambda^{(0)I}\Lambda^{(0)J}+4\bar{U}bf\left(i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J-\boldsymbol{L}\chi+\bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}}\right)f\bar{b}U,\end{aligned} \tag{A.15}$$

where $\boldsymbol{L}\chi$ is defined by

$$\boldsymbol{L}\chi=\frac{1}{2}\left\{\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}},\chi\right\}+\left[a'_\mu,[a'^\mu,\chi]\right]. \tag{A.16}$$

Then $\varphi^{(0)}$ satisfies the equation of motion if χ satisfies

$$\boldsymbol{L}\chi=i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J+\bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}}. \tag{A.17}$$

In the case of $C^{\mu\nu}\neq 0$, the ansatz of the solution is changed as [25]

$$\varphi^{(0)}=-i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{U}\mathcal{M}^I f\bar{\mathcal{M}}^J U+\bar{U}\begin{pmatrix}\phi & 0\\ 0 & \chi\mathbf{1}_2+\mathbf{1}_k C\end{pmatrix}U, \tag{A.18}$$

where C is the 2×2 matrix of which component is $C_\alpha{}^\beta=\frac{1}{2}C^{\mu\nu}(\sigma_{\mu\nu})_\alpha{}^\beta$. Now one can show that

$$\begin{aligned}\nabla^2\varphi^{(0)}=-i\sqrt{2}\epsilon_{IJ}\Lambda^{(0)I}\Lambda^{(0)J}+4\bar{U}bf\Bigg[&i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J\\ &-2f^{-1}C-\{f^{-1},\chi\}+\bar{\Delta}^{\dot{\alpha}}\begin{pmatrix}\phi & 0\\ 0 & \chi\mathbf{1}_2+\mathbf{1}_k C\end{pmatrix}\Delta_{\dot{\alpha}}\Bigg]f\bar{b}U.\end{aligned} \tag{A.19}$$

The third term in the right hand side of (A.19) becomes the deformation term in the equation of motion (A.13) due to (A.7). The C-dependent part in the last term is

$$\bar{\Delta}^{\dot{\alpha}}\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}_k C\end{pmatrix}\Delta_{\dot{\alpha}}=(\bar{a}'+\bar{x})^{\dot{\alpha}\alpha}C_\alpha{}^\beta(a'+x)_{\beta\dot{\alpha}}=C^{\mu\nu}[a'_\mu,a'_\nu]. \tag{A.20}$$

Then we obtain

$$\nabla^2\varphi^{(0)} = -i\sqrt{2}\epsilon_{IJ}\Lambda^{(0)I}\Lambda^{(0)J} - iC^{\mu\nu}F^{(0)}_{\mu\nu} + 4\bar{U}bf\left(i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J - \boldsymbol{L}\chi + \bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}} + C^{\mu\nu}[a'_\mu, a'_\nu]\right)f\bar{b}U. \tag{A.21}$$

Hence $\varphi^{(0)}$ is the solution of the deformed equation of motion if χ satisfies

$$\boldsymbol{L}\chi = i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J + \bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}} + C^{\mu\nu}[a'_\mu, a'_\nu]. \tag{A.22}$$

We can also solve the instanton equation of $\bar{\varphi}^{(0)}$ in a similar way.

B Detailed calculations of the instanton effective action

B.1 Calculation of (2.58) and (2.59)

Here we give the detail for the calculation of (2.58) and (2.59). In order to calculate (2.58), we use the formula

$$\left[\bar{\varphi}^{(0)}, \Lambda^{(0)}_{\alpha I}\right] = (\sigma^\mu)_{\alpha\dot{\alpha}}\nabla_\mu\bar{\psi}^{\dot{\alpha}}_I + \Lambda_\alpha(\mathcal{N}_I) + \epsilon_{IJ}\bar{C}_\alpha{}^\beta\Lambda_\beta(\mathcal{M}^J), \tag{B.1}$$

where $\bar{\psi}_I$ and $\mathcal{N}_I$ are given by

$$\bar{\psi}^{\dot{\alpha}}_I = \bar{\psi}^{(1)\dot{\alpha}}_I + \bar{\psi}^{(2)\dot{\alpha}}_I, \tag{B.2}$$

$$\bar{\psi}^{(1)\dot{\alpha}}_I = \frac{1}{2}\bar{U}\left[-\mathcal{M}_I f\bar{\Delta}^{\dot{\alpha}}\begin{pmatrix}\bar{\phi} & 0\\ 0 & \bar{\chi}\mathbf{1}_2 + \mathbf{1}_k\bar{C}\end{pmatrix} + \begin{pmatrix}\bar{\phi} & 0\\ 0 & \bar{\chi}\mathbf{1}_2 + \mathbf{1}_k\bar{C}\end{pmatrix}\Delta^{\dot{\alpha}}f\bar{\mathcal{M}}_I\right]U, \tag{B.3}$$

$$\bar{\psi}^{(2)\dot{\alpha}}_I = \bar{U}\begin{pmatrix}0 & 0\\ 0 & \mathcal{G}^{\dot{\alpha}}_I\mathbf{1}_2\end{pmatrix}U, \quad \partial_\mu\mathcal{G}^{\dot{\alpha}}_I = 0, \tag{B.4}$$

$$\mathcal{N}_I = \begin{pmatrix}\bar{\phi} & 0\\ 0 & \bar{\chi}\mathbf{1}_2 + \mathbf{1}_k\bar{C}\end{pmatrix}\mathcal{M}_I - \mathcal{M}_I\bar{\chi} + 2\begin{pmatrix}0 & 0\\ 0 & \mathcal{G}^{\dot{\alpha}}_I\end{pmatrix}a_{\dot{\alpha}} - 2a_{\dot{\alpha}}\mathcal{G}^{\dot{\alpha}}_I. \tag{B.5}$$

Here the $k \times k$ matrix $\mathcal{G}_I^{\dot{\alpha}}$ is chosen such that $\mathcal{N}_I$ satisfies the fermionic ADHM condition $\bar{\mathcal{N}}_I \Delta^{\dot{\alpha}} + \bar{\Delta}^{\dot{\alpha}} \mathcal{N}_I = 0$. From the formula (B.1), (2.58) becomes

$$\int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[-\frac{i}{\sqrt{2}} \Lambda^{(0)\alpha I} \big[\bar{\varphi}^{(0)}, \Lambda^{(0)}_{\alpha I}\big] - \frac{i}{\sqrt{2}} \bar{C}^{(\alpha\beta)} \Lambda^{(0)I}_{\alpha} \Lambda^{(0)}_{\beta I}\Big]$$
$$= \int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[-\frac{i}{\sqrt{2}} \partial_\mu \big(\Lambda(\mathcal{M}^I) \sigma^\mu \bar{\psi}_I\big) - \frac{i}{\sqrt{2}} \Lambda^\alpha(\mathcal{M}^I) \Lambda_\alpha(\mathcal{N}_I)\Big]. \tag{B.6}$$

The first term in the right hand side of (B.6) vanishes since $\Lambda(\mathcal{M}^I)\sigma^\mu \bar{\psi}_I$ behaves as $|x|^{-5}$ for large $|x|$. The second term can be evaluated using Corrigan's inner-product formula [36, 37]

$$\int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[\Lambda^\alpha(\mathcal{M}^I) \Lambda_\alpha(\mathcal{N}_I)\Big] = -\frac{\pi^2}{2\kappa} \mathrm{tr}_k \Big[\bar{\mathcal{M}}^I (\mathcal{P}_\infty + 1) \mathcal{N}_I + \bar{\mathcal{N}}_I (\mathcal{P}_\infty + 1) \mathcal{M}^I\Big]$$
$$= -\frac{2\pi^2}{\kappa} \epsilon_{IJ} \mathrm{tr}_k \Big[\bar{\mu}^I \bar{\phi} \mu^J - \bar{\mathcal{M}}^I \mathcal{M}^J \bar{\chi} + \frac{1}{2} \bar{C}^{(\alpha\beta)} \mathcal{M}'^I_\alpha \mathcal{M}'^J_\beta\Big], \tag{B.7}$$

where $\mathcal{P}_\infty = \lim_{|x|\to\infty} U\bar{U}$. Since the part proportional to $\mathcal{G}_I^{\dot{\alpha}}$ vanishes in (B.7) by fermionic ADHM condition (2.18), we do not need to solve $\mathcal{G}_I^{\dot{\alpha}}$ explicitly. Then we obtain (2.58)

$$\int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[-\frac{i}{\sqrt{2}} \Lambda^{(0)\alpha I} \big[\bar{\varphi}^{(0)}, \Lambda^{(0)}_{\alpha I}\big] - \frac{i}{\sqrt{2}} \bar{C}^{(\alpha\beta)} \Lambda^{(0)I}_{\alpha} \Lambda^{(0)}_{\beta I}\Big]$$
$$= \frac{1}{\kappa} \sqrt{2} \pi^2 i \epsilon_{IJ} \mathrm{tr}_k \Big[\bar{\mu}^I \bar{\phi} \mu^J - \bar{\mathcal{M}}^I \mathcal{M}^J \bar{\chi} + \frac{1}{2} \bar{C}^{(\alpha\beta)} \mathcal{M}'^I_\alpha \mathcal{M}'^J_\beta\Big]. \tag{B.8}$$

Nextly we prove (2.59). The left hand side in (2.59) can be rewritten as

$$\int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[-i\bar{\varphi}^{(0)} F^{(0)}_{\mu\nu} C^{\mu\nu}\Big]$$
$$= -8 C_\alpha{}^\beta \int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\left[\bar{U} \begin{pmatrix} \bar{\phi} & 0 \\ 0 & \bar{\chi} \mathbf{1}_2 + \mathbf{1}_k \bar{C} \end{pmatrix} \mathcal{P} b^\alpha f \bar{b}_\beta U\right]$$
$$= -2 C_\alpha{}^\beta \int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[(\sigma^{\mu\nu})_\beta{}^\alpha \nabla_\mu R_\nu + 2\bar{U} b^\alpha \{f, \bar{\chi}\} \bar{b}_\beta U\Big]. \tag{B.9}$$

Here R_μ is defined by

$$R_\mu = \frac{1}{2} \bar{U} \left[b^\alpha (\sigma_\mu)_{\alpha\dot{\alpha}} f \bar{\Delta}^{\dot{\alpha}} \begin{pmatrix} \bar{\phi} & 0 \\ 0 & \bar{\chi} \mathbf{1}_2 + \mathbf{1}_k \bar{C} \end{pmatrix} - \begin{pmatrix} \bar{\phi} & 0 \\ 0 & \bar{\chi} \mathbf{1}_2 + \mathbf{1}_k \bar{C} \end{pmatrix} \Delta_{\dot{\alpha}} (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} f \bar{b}_\alpha\right] U. \tag{B.10}$$

Then the first term of (B.9) is evaluated as

$$-2C_\alpha{}^\beta \int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[(\sigma^{\mu\nu})_\beta{}^\alpha \nabla_\mu R_\nu\Big] = 2C^{\mu\nu} \int d^4x \, \frac{1}{\kappa} \partial_\mu \mathrm{Tr} R_\nu$$
$$= \frac{1}{\kappa} \pi^2 C^{\mu\nu} \bar{C}_{\mu\nu} \mathrm{tr}_k \big[\bar{w}^{\dot\alpha} w_{\dot\alpha}\big]. \qquad \text{(B.11)}$$

The second term of (B.9) becomes a total derivative and is calculated as

$$-4C_\alpha{}^\beta \int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[\bar{U} b^\alpha \{f, \bar\chi\} \bar{b}_\beta U\Big] = \frac{4}{\kappa} C^{\mu\nu} \int d^4x \, \partial_\mu \mathrm{tr}_k \Big[\big[f, a'_\nu\big] \bar\chi\Big]$$
$$= -\frac{4\pi^2}{\kappa} \mathrm{tr}_k \Big[C^{\mu\nu} [a'_\mu, a'_\nu] \bar\chi\Big], \qquad \text{(B.12)}$$

where we have used the asymptotic behavior of f given by

$$f = \frac{1}{|x|^2} \mathbf{1}_k - \frac{2x^\lambda}{|x|^4} a'_\lambda + \mathcal{O}(|x|^{-4}). \qquad \text{(B.13)}$$

Finally, we obtain the result in (2.59)

$$\int d^4x \, \frac{1}{\kappa} \mathrm{Tr}\Big[-i\bar\varphi^{(0)} F^{(0)}_{\mu\nu} C^{\mu\nu}\Big] = \frac{\pi^2}{\kappa} \mathrm{tr}_k \Big[-4C^{\mu\nu} [a'_\mu, a'_\nu] \bar\chi + C^{\mu\nu} \bar{C}_{\mu\nu} \bar{w}^{\dot\alpha} w_{\dot\alpha}\Big]. \qquad \text{(B.14)}$$

B.2 Verification of the form of the discrepancy term

The contribution from (2.66) to the instanton effective action , $\delta S^{(0)}_{\mathrm{eff}}$ is

$$\delta S^{(0)}_{\mathrm{eff}} = -\frac{1}{16\kappa} \int d^4x \; C^{\rho\sigma} \bar{C}_{\rho\sigma} |x|^2 \mathrm{Tr}\left[F^{(0)\mu\nu} F^{(0)}_{\mu\nu}\right]. \qquad \text{(B.15)}$$

From Osborn's formula [38] $\mathrm{Tr}\left[F^{(0)\mu\nu} F^{(0)}_{\mu\nu}\right] = -\Box^2 \mathrm{tr}_k \log f$, $\delta S^{(0)}_{\mathrm{eff}}$ can be rewritten in a total derivative as

$$\begin{aligned} \delta S^{(0)}_{\mathrm{eff}} &= \frac{1}{16\kappa} C^{\rho\sigma} \bar{C}_{\rho\sigma} \int d^4x \; |x|^2 \Box^2 \mathrm{tr}_k \log f \\ &= \frac{1}{16\kappa} C^{\rho\sigma} \bar{C}_{\rho\sigma} \int d^4x \; \partial_\mu \left[|x|^2 \Box \partial^\mu - 2x^\mu \Box + 8\partial^\mu\right] \mathrm{tr}_k \log f. \end{aligned} \qquad \text{(B.16)}$$

Plugging the explicit form of f (A.3) into (B.16), we obtain (2.64).

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